

# ADELIC VERSIONS OF THE WEIERSTRASS APPROXIMATION THEOREM

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**ABSTRACT.** Let  $\underline{E} = \prod_{p \in \mathbb{P}} E_p$  be a compact subset of  $\widehat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$  and denote by  $\mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$  the ring of continuous functions from  $\underline{E}$  into  $\widehat{\mathbb{Z}}$ . We obtain two kinds of adelic versions of the Weierstrass approximation theorem. Firstly, we prove that the ring  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) := \{f(x) \in \mathbb{Q}[x] \mid \forall p \in \mathbb{P}, f(E_p) \subseteq \mathbb{Z}_p\}$  is dense in the direct product  $\prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p)$  for the uniform convergence topology. Secondly, under the hypothesis that, for each  $n \geq 0$ ,  $\#(E_p \pmod{p}) > n$  for all but finitely many  $p$ , we prove the existence of regular bases of the  $\mathbb{Z}$ -module  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ , and show that, for such a basis  $\{f_n\}_{n \geq 0}$ , every function  $\varphi$  in  $\prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p)$  may be uniquely written as a series  $\sum_{n \geq 0} \mathcal{L}_n f_n$  where  $\mathcal{L}_n \in \widehat{\mathbb{Z}}$  and  $\lim_{n \rightarrow \infty} \mathcal{L}_n \rightarrow 0$ .

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## 1. INTRODUCTION:

### THE CLASSICAL $p$ -ADIC VERSIONS OF WEIERSTRASS THEOREM

A well known  $p$ -adic version of the Weierstrass polynomial approximation theorem due to Dieudonné [9] states the following:

**Theorem 1.1.** *For every compact subset  $E$  of  $\mathbb{Q}_p$ , the ring of polynomial functions  $\mathbb{Q}_p[x]$  is dense in the ring  $\mathcal{C}(E, \mathbb{Q}_p)$  of continuous functions from  $E$  into  $\mathbb{Q}_p$  with respect to the uniform convergence topology.*

If we restrict to functions with values in  $\mathbb{Z}_p$ , we have to consider the subring of polynomial functions whose values on  $E$  are in  $\mathbb{Z}_p$ , namely the ring  $\text{Int}(E, \mathbb{Z}_p) = \{f \in \mathbb{Q}_p[x] \mid f(E) \subseteq \mathbb{Z}_p\}$ , and then, following Kaplansky [10], we have:

**Theorem 1.2.** *For every compact subset  $E$  of  $\mathbb{Q}_p$ , the ring of polynomial functions  $\text{Int}(E, \mathbb{Z}_p)$  is dense in  $\mathcal{C}(E, \mathbb{Z}_p)$ , the ring of continuous functions from  $E$  into  $\mathbb{Z}_p$  with respect to the uniform convergence topology.*

In the particular case where  $E$  is equal to  $\mathbb{Z}_p$ , Mahler [11] gave an effective approximation theorem in terms of the binomial polynomials  $\binom{x}{n} := \frac{x(x-1)\cdots(x-(n-1))}{n!}$ .

**Theorem 1.3.** *Every continuous function  $\varphi \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Q}_p)$  may be uniquely written as a series in the binomial polynomials  $\binom{x}{n}$  with coefficients  $c_n$  in  $\mathbb{Q}_p$ :*

$$\varphi(x) = \sum_{n \geq 0} c_n \binom{x}{n}, \text{ where } v_p(c_n) \rightarrow +\infty \text{ and } \inf_{x \in \mathbb{Z}_p} v_p(\varphi(x)) = \inf_{n \geq 0} v_p(c_n).$$

It is straightforward to see that, if  $\varphi \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$ , then the coefficients  $c_n$ 's lie in  $\mathbb{Z}_p$  and the partial sums  $\sum_{n \geq 0}^d c_n \binom{x}{n}$  are elements of  $\text{Int}(\mathbb{Z}_p, \mathbb{Z}_p)$  (denoted by  $\text{Int}(\mathbb{Z}_p)$ ), so we find again Kaplansky's Theorem when  $E = \mathbb{Z}_p$ . Mahler's result was extended to regular compact subsets of  $\mathbb{Q}_p$  by Amice [1], and then to every compact subset of  $\mathbb{Q}_p$  by Bhargava and Kedlaya [4]:

**Theorem 1.4.** *For each compact subset  $E$  of  $\mathbb{Q}_p$ , there exists a sequence  $\{a_n\}_{n \geq 0}$  of elements of  $E$  such that the polynomials  $f_n(x) = \prod_{k=0}^{n-1} \frac{x-a_k}{a_n-a_k}$ ,  $n \in \mathbb{N}$ , form a basis of the  $\mathbb{Z}_p$ -module  $\text{Int}(E, \mathbb{Z}_p)$ . Then, every  $\varphi \in \mathcal{C}(E, \mathbb{Q}_p)$  may be uniquely written as a series in the  $f_n$ 's with coefficients in  $\mathbb{Q}_p$ :*

$$\varphi(x) = \sum_{n \geq 0} c_n f_n(x) \text{ where } c_n \in \mathbb{Q}_p \text{ and } \lim_{n \rightarrow +\infty} v_p(c_n) = +\infty.$$

Moreover, one knows that with the previous notation:

$$\inf_{x \in E} v_p(\varphi(x)) = \inf_{n \geq 0} v_p(c_n).$$

Our aim is to obtain adelic versions of these results. In the next section we recall the restricted product topology on the ring  $\mathcal{A}_f(\mathbb{Q})$  of finite adèles of  $\mathbb{Q}$  since we are interested in the Banach space of continuous functions  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$  where  $\underline{E}$  denotes a compact subset of  $\mathcal{A}_f(\mathbb{Q})$ . The subring of polynomial functions that we will consider first is the ring  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) := \{f \in \mathbb{Q}[x] \mid f(\underline{E}) \subseteq \widehat{\mathbb{Z}}\}$ . In section 3, we establish the following theorem:

**Theorem 1.5.** *Let  $\underline{E} = \prod_{p \in \mathbb{P}} E_p$  be a compact subset of  $\widehat{\mathbb{Z}}$ . The topological closure (with respect to the uniform convergence topology) of the ring of polynomial functions  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  in the ring  $\mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$  of continuous functions from  $\underline{E}$  into  $\widehat{\mathbb{Z}}$  is the ring  $\prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p)$ .*

This statement is noteworthy in so far as a polynomial with a single variable can simultaneously approach several functions (of one variable but on subsets of distinct  $\mathbb{Z}_p$ 's). Then, in order to extend Bhargava and Kedlaya's result, we characterize in section 4 the subsets  $\underline{E}$  of  $\widehat{\mathbb{Z}}$  such that the  $\mathbb{Z}$ -module  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  admits bases formed by one polynomial of each degree (called *regular bases*), and then, we show how we can construct such a basis provided they exist. But this not enough: in order to obtain the adelic analogue of Theorem 1.4 we have to introduce in section 5 polynomials whose coefficients are adèles and we consider bases of the  $\widehat{\mathbb{Z}}$ -module  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ . Finally, in section 6, we use all the previous results to obtain an analogue of Bhargava and Kedlaya's result:

**Theorem 1.6.** *Let  $\underline{E} = \prod_{p \in \mathbb{P}} E_p$  be a compact subset of  $\widehat{\mathbb{Z}}$  such that all  $E_p$ 's are infinite and, for each  $n \geq 0$ ,  $\#(E_p \pmod{p}) > n$  for all but finitely many  $p$ . Then, there exist bases of the  $\mathbb{Z}$ -module  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ , and, for such a basis  $\{f_n\}_{n \geq 0}$ , every  $\varphi \in \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p)$  may be uniquely written as a series  $\sum_{n \geq 0} \underline{c}_n f_n$  where  $\underline{c}_n \in \widehat{\mathbb{Z}}$  and  $\lim_{n \rightarrow \infty} \underline{c}_n \rightarrow 0$ .*

## 2. THE ADELIC FRAMEWORK

Let  $\mathcal{A}_f(\mathbb{Q})$  denote the topological ring of finite adèles of  $\mathbb{Q}$ . Recall that, as a subset,  $\mathcal{A}_f(\mathbb{Q})$  is the restricted product of the fields  $\mathbb{Q}_p$  with respect to the rings

$\mathbb{Z}_p$ , as  $p$  ranges through the set of primes  $\mathbb{P}$ , that is:

$$\mathcal{A}_f(\mathbb{Q}) = \left\{ \underline{x} = (x_p) \in \prod_{p \in \mathbb{P}} \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for all but finitely many } p \in \mathbb{P} \right\}.$$

The topology on  $\mathcal{A}_f(\mathbb{Q})$  is the restricted product topology characterized by the fact that the following subsets form a basis of open subsets:  $\prod_{p \in \mathbb{P}} O_p$  where  $O_p$  is an open subset of  $\mathbb{Q}_p$  and  $O_p = \mathbb{Z}_p$  for all but finitely many  $p$ .

Clearly,  $\mathcal{A}_f(\mathbb{Q})$  is locally compact and contains the compact subring  $\widehat{\mathbb{Z}}$ , the profinite completion of  $\mathbb{Z}$  with respect to the ideal topology, that is,

$$\widehat{\mathbb{Z}} = \varprojlim_{n \geq 1} \mathbb{Z}/n\mathbb{Z} \cong \prod_{p \in \mathbb{P}} \mathbb{Z}_p.$$

Note also that  $\mathbb{Q}$  may be embedded in  $\mathcal{A}_f(\mathbb{Q})$  in the following way:

$$(1) \quad j : \mathbb{Q} \ni \alpha \mapsto j(\alpha) = (\alpha_p)_{p \in \mathbb{P}} \in \mathcal{A}_f(\mathbb{Q}), \text{ where } \alpha_p = \alpha, \forall p \in \mathbb{P}.$$

In the sequel,  $\underline{E}$  will always denote a compact subset of  $\mathcal{A}_f(\mathbb{Q})$ . For every compact subset  $\underline{E}$  of  $\mathcal{A}_f(\mathbb{Q})$ , we consider the  $\mathbb{Q}$ -vector space  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$  of continuous functions from  $\underline{E}$  into  $\mathcal{A}_f(\mathbb{Q})$ .

The topology on  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$  that we will consider is the uniform convergence topology with respect to the restricted product topology on  $\mathcal{A}_f(\mathbb{Q})$ . Endowed with this topology, the  $\mathbb{Q}$ -vector space  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$  is a Banach space.

One might be tempted to find some ring of polynomial functions with rational coefficients which could be dense in  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$ . In fact, our first result is negative.

**Proposition 2.1.** *The Banach space  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$  does not contain any dense  $\mathbb{Q}$ -subspace of polynomial functions.*

*Proof.* It is enough to prove that even one component of one function restricted to one variable cannot be approximated. Thus, we will prove that  $\mathbb{Q}[x]$  is not dense in  $\mathcal{C}(\mathbb{Z}_q, \mathbb{Z}_p)$  where  $p \neq q \in \mathbb{P}$ . Indeed, the function  $\chi : \mathbb{Z}_q \rightarrow \mathbb{Z}_p$  which is the characteristic function of the subset  $q\mathbb{Z}_q$  is continuous. Assume that  $f \in \mathbb{Q}[x]$  is an approximation of  $\chi \bmod p\mathbb{Z}_p$ , that is,  $f(q\mathbb{Z}_p) \subseteq 1 + p\mathbb{Z}_p$  and  $f(\mathbb{Z}_q \setminus q\mathbb{Z}_q) \subseteq p\mathbb{Z}_p$ . In particular,  $f(0) \in 1 + p\mathbb{Z}_p$  while  $f(p^k) \in p\mathbb{Z}_p$  for all  $k \geq 1$ . The fact that, for every polynomial  $g \in \mathbb{Q}[x]$ ,  $g(p^k) - g(0) \in p\mathbb{Z}_p$  for  $k$  large enough leads to a contradiction.  $\square$

Thus, the topological closure of subrings of polynomial functions in  $\mathbb{Q}[x]$  will be strictly contained in  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$ . We continue with some remarks about compact subsets and continuous functions.

*Remarks 2.2.* Denote by  $\pi_p$  the canonical surjection from  $\mathcal{A}_f(\mathbb{Q})$  onto  $\mathbb{Q}_p$ .

(1) Since  $\underline{E}$  is a compact subset of  $\mathcal{A}_f(\mathbb{Q})$ , the projections  $\pi_p(\underline{E})$  of  $\underline{E}$  are compact subsets of  $\mathbb{Q}_p$ , and hence,  $\underline{E}$  is contained in the compact subset  $\prod_{p \in \mathbb{P}} \pi_p(\underline{E})$ . Moreover, by definition of the topology on  $\mathcal{A}_f(\mathbb{Q})$ , we necessarily have that  $\pi_p(\underline{E}) \subseteq \mathbb{Z}_p$  for all but finitely many  $p$ . In other words, there exists  $d \in \mathbb{N}^*$  such that  $d\underline{E} \subseteq \widehat{\mathbb{Z}}$ .

(2) For every function  $\underline{\psi}$  from  $\underline{E}$  into  $\mathcal{A}_f(\mathbb{Q})$ , we denote by  $\psi_p$  its components:  $\psi_p = \pi_p \circ \underline{\psi} : \underline{E} \rightarrow \mathbb{Q}_p$ . Clearly, if  $\underline{\psi} \in \mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$ , then the components  $\psi_p$  are continuous and belong to  $\mathcal{C}(\underline{E}, \mathbb{Q}_p)$ .

(3) Since the subset  $\underline{E}$  is compact, every continuous function  $\underline{\psi} \in \mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$  is uniformly continuous and its image  $\underline{\psi}(\underline{E})$  is a compact subset of  $\mathcal{A}_f(\mathbb{Q})$ . Consequently, for every  $\underline{\psi} \in \mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$ , there exists  $d_1 \in \mathbb{N}^*$  such that  $d_1 \cdot \underline{\psi}(\underline{E}) \subseteq \widehat{\mathbb{Z}}$ , that is,  $d_1 \underline{\psi} \in \mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$ .

(4) Finally, for every  $\underline{\psi} \in \mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$ , there exist  $d$  and  $d_1 \in \mathbb{N}^*$  such  $d \cdot \underline{E} \subseteq \widehat{\mathbb{Z}}$  and  $d_1 \cdot \underline{\psi}(\underline{E}) \subseteq \widehat{\mathbb{Z}}$ . Then, the function  $\underline{\varphi}$  defined by  $\underline{\varphi}(\underline{x}) = d_1 \underline{\psi}(\frac{1}{d} \underline{x})$  belongs to  $\mathcal{C}(d\underline{E}, \widehat{\mathbb{Z}})$ . Thus, in order to prove our approximation theorems, without loss of generality we will consider functions  $\varphi \in \mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$  where  $\underline{E}$  is a compact subset of  $\widehat{\mathbb{Z}}$ .

Now, the question could be: which subring of polynomial functions shall we consider? First, we make the choice of considering polynomials in one variable with coefficients in  $\mathbb{Q}$ . We will see why this choice is sufficient at least for the first part of our work.

For  $f = \sum_{k=0}^n c_k x^k \in \mathbb{Q}[x]$  and  $\underline{\alpha} = (\alpha_p) \in \mathcal{A}_f(\mathbb{Q})$ , the value of  $f$  at  $\underline{\alpha}$  is:

$$f(\underline{\alpha}) = \sum_{k=0}^n c_k \underline{\alpha}^k = \left( \sum_{k=0}^n c_k \alpha_p^k \right)_{p \in \mathbb{P}} = (f(\alpha_p))_{p \in \mathbb{P}} \in \prod_{p \in \mathbb{P}} \mathbb{Q}_p.$$

Note that these equalities follow from the structure of  $\mathcal{A}_f(\mathbb{Q})$  as  $\mathbb{Z}$ -algebra. Clearly, we have  $f(\underline{\alpha}) \in \mathcal{A}_f(\mathbb{Q})$ . Thus, every  $f \in \mathbb{Q}[x]$  may be considered as a function from  $\mathcal{A}_f(\mathbb{Q})$  into itself defined by:

$$(2) \quad \forall f \in \mathbb{Q}[x], \quad \mathcal{A}_f(\mathbb{Q}) \ni \underline{\alpha} = (\alpha_p)_{p \in \mathbb{P}} \mapsto f(\underline{\alpha}) = (f(\alpha_p))_{p \in \mathbb{P}} \in \mathcal{A}_f(\mathbb{Q}).$$

Are they continuous functions? Fix some  $\underline{\alpha} \in \mathcal{A}_f(\mathbb{Q})$  and some basic open neighborhood  $\underline{Q} = \prod_{p \in \mathbb{P}} O_p$  of 0 in  $\mathcal{A}_f(\mathbb{Q})$ . Clearly the components of  $f$  (which are all equal to  $f$ ) are continuous functions from  $\mathbb{Q}_p$  into  $\mathbb{Q}_p$ , and hence, for every  $p \in \mathbb{P}$ , there exists an open neighborhood  $V_p$  of 0 in  $\mathbb{Q}_p$  such that  $(\beta_p - \alpha_p) \in V_p$  implies  $(f(\beta_p) - f(\alpha_p)) \in O_p$ . Let  $\mathbb{P}_1$  be the subset of  $\mathbb{P}$  formed by the  $p$ 's such that  $f \in \mathbb{Z}_p[x]$ ,  $O_p = \mathbb{Z}_p$ , and  $\alpha_p \in \mathbb{Z}_p$ . For  $p \in \mathbb{P}_1$ , we may choose  $\mathbb{Z}_p$  instead of  $V_p$ , and then, since  $\mathbb{P} \setminus \mathbb{P}_1$  is finite, we may consider the basic open neighborhood of 0 :  $\underline{V} = \prod_{p \in \mathbb{P}_1} \mathbb{Z}_p \times \prod_{p \notin \mathbb{P}_1} V_p$  which is such that  $(\underline{\beta} - \underline{\alpha}) \in \underline{V} \Rightarrow (f(\underline{\beta}) - f(\underline{\alpha})) \in \underline{Q}$ . Finally,  $f$  is a uniformly continuous function from  $\mathcal{A}_f(\mathbb{Q})$  into itself.

In other words, with the previous convention (2) about the polynomial functions  $f$ , we have the following containment:

$$(3) \quad \forall \underline{E} \subseteq \mathcal{A}_f(\mathbb{Q}), \quad \mathbb{Q}[x] \subset \mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q})).$$

Now, we may state our adelic theorems.

### 3. FIRST ADELIC WEIERSTRASS THEOREMS

For simplicity, we consider first the ring of continuous functions  $\mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$  where  $\underline{E}$  a compact subset of  $\widehat{\mathbb{Z}}$  of the form  $\underline{E} = \prod_{p \in \mathbb{P}} E_p$  where each  $E_p$  is a compact subset of the ring of  $p$ -adic integers  $\mathbb{Z}_p$ , for each prime  $p \in \mathbb{Z}$ . Note that the topology induced on  $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$  by the restricted product topology of  $\mathcal{A}_f(\mathbb{Q})$  is nothing else than the product topology.

The subring of  $\mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$  formed by the rational polynomial functions, that is,  $\mathcal{C}(\underline{E}, \widehat{\mathbb{Z}}) \cap \mathbb{Q}[x]$  will be denoted by  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ .

**Definition 3.1.** The ring of rational polynomials which are *integer valued on the subset  $\underline{E}$  of  $\widehat{\mathbb{Z}}$*  is the ring  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) = \{f \in \mathbb{Q}[x] \mid f(\underline{E}) \subseteq \widehat{\mathbb{Z}}\}$ , that is,

$$\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) := \{f \in \mathbb{Q}[x] \mid \forall \underline{\alpha} = (\alpha_p)_p \in \underline{E}, \forall p \in \mathbb{P}, f(\alpha_p) \in \mathbb{Z}_p\}.$$

This notation is a particular case of the following that we will use in the whole paper: if  $\mathcal{A}$  is a commutative ring,  $A$  and  $B$  are subrings of  $\mathcal{A}$  and  $E$  is a subset of  $\mathcal{A}$ , then

$$\text{Int}_A(E, B) := \{f \in A[x] \mid f(E) \subseteq B\}.$$

When the ring  $B$  is an integral domain and  $A$  denotes its quotient field, we will omit the index  $A$ , for example as with the previous notation  $\text{Int}(E, \mathbb{Z}_p)$  where  $\mathbb{Q}_p$  is omitted. Now, we can state the adelic analogue of Kaplansky's result (Theorem 1.2), which we restate here for the sake of the reader.

**Theorem 3.2.** Let  $\underline{E} = \prod_{p \in \mathbb{P}} E_p$  be a compact subset of  $\widehat{\mathbb{Z}}$ . The topological closure (with respect to the uniform convergence topology) of the ring of polynomial functions  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  in the ring  $\mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$  of continuous functions from  $\underline{E}$  into  $\widehat{\mathbb{Z}}$  is the ring  $\prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p)$ .

*Proof.* Assume that  $\varphi = (\varphi_p)_{p \in \mathbb{P}} \in \mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$  belongs to the topological closure of  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ . Fix some  $p \in \mathbb{P}$ . Then, whatever  $k \in \mathbb{N}$ , there exists  $f \in \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  such that, for every  $\underline{\alpha} \in \underline{E}$ ,  $v_p(\varphi_p(\underline{\alpha}) - f_p(\underline{\alpha})) \geq k$ . The fact that  $f_p(\underline{\alpha}) = f(\alpha_p)$  implies that, for all  $\underline{\alpha}$  and  $\underline{\beta} \in \underline{E}$  such that  $\alpha_p = \beta_p$ , one has  $v_p(\varphi_p(\underline{\alpha}) - \varphi_p(\underline{\beta})) \geq k$ . Since  $k$  may be as large as we want, we may conclude that  $\alpha_p = \beta_p$  implies  $\varphi_p(\underline{\alpha}) = \varphi_p(\underline{\beta})$ . Thus,  $\varphi_p(\underline{\alpha})$  is a function of  $\alpha_p$  only, in other words,  $\varphi_p \in \mathcal{C}(E_p, \mathbb{Z}_p)$ . Finally, the topological closure of  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  in  $\mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$  is contained in  $\prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p)$ .

Let us prove the reverse containment. Let  $\varphi = (\varphi_p)_{p \in \mathbb{P}}$  be any element of  $\prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p)$  and let  $\underline{Q} = \prod_{p \in \mathbb{P}} O_p$  be any basic open neighborhood of 0 in  $\widehat{\mathbb{Z}}$ . If  $p \in \mathbb{P}$  is such that  $O_p \neq \mathbb{Z}_p$  (they are at most finitely many), let  $k_p \geq 0$  be such that  $p^{k_p} \mathbb{Z}_p \subseteq O_p$ . By Theorem 1.4 there exists a polynomial function  $f_p \in \text{Int}(E_p, \mathbb{Z}_p)$  such that  $v_p(\varphi_p(\alpha_p) - f_p(\alpha_p)) \geq k_p$  for every  $\alpha_p \in E_p$ . For all these  $p$ , write  $f_p(x) = \sum_r c_{p,r} x^r$  with  $c_{p,r} \in \mathbb{Q}_p$ . By an extension of the Chinese remainder theorem, for each  $r \geq 0$ , there exists  $c_r \in \mathbb{Q}$  such that  $v_p(c_r - c_{p,r}) \geq \sup_{p \in \mathbb{P}} k_p$  if  $O_p \neq \mathbb{Z}_p$  and  $v_p(c_r) \geq 0$  if  $O_p = \mathbb{Z}_p$ . Then, for every  $p \in \mathbb{P}$ , the polynomial  $f(x) = \sum c_r x^r \in \mathbb{Q}[x]$  satisfies:

$$\begin{aligned} &v_p(\varphi_p(\alpha_p) - f(\alpha_p)) \geq \\ &\begin{cases} \inf\{v_p(\varphi_p(\alpha_p) - f_p(\alpha_p)), v_p(f_p(\alpha_p) - f(\alpha_p))\} \geq k_p & \text{if } O_p \neq \mathbb{Z}_p \\ \inf\{v_p(\varphi_p(\alpha_p)), v_p(f(\alpha_p))\} \geq 0 & \text{if } O_p = \mathbb{Z}_p \end{cases} \end{aligned}$$

Thus,  $\varphi(\underline{\alpha}) - f(\underline{\alpha}) \in \underline{Q}$  for every  $\underline{\alpha} \in \underline{E}$ . Moreover, clearly,  $f \in \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ .  $\square$

*Remarks 3.3.* (1) At the end of the proof of Theorem 3.2, we used a kind of Chinese remainder theorem among the different overrings  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$ , which in fact are localizations of  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  with respect to  $p$  (see formula (5) below), namely:

Fix a finite set of primes  $P_0 = \{p_i \in \mathbb{P} \mid 1 \leq i \leq r\}$  and a corresponding set of positive integers  $\{k_i \in \mathbb{N}^* \mid 1 \leq i \leq r\}$ . Whatever  $f_i \in \text{Int}_{\mathbb{Q}}(E_{p_i}, \mathbb{Z}_{p_i})$  for  $i = 1, \dots, r$ , there exists  $f \in \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  such that  $f \equiv f_i \pmod{p_i^{k_i}}$  for all  $p_i \in P_0$  and  $f \in \mathbb{Z}_{(p)}[x]$  for all  $p \in \mathbb{P} \setminus P_0$ .

Here,  $f \equiv g \pmod{p^k}$  means that  $f - g \in p^k \mathbb{Z}_{(p)}[x]$ .

(2) If  $\underline{E}$  denotes any compact subset of  $\widehat{\mathbb{Z}}$ , the statement of Theorem 3.2 could be: the ring  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  is dense in the ring  $\prod_{p \in \mathbb{P}} \mathcal{C}(\pi_p(\underline{E}), \mathbb{Z}_p)$  for the uniform convergence topology associated to the restricted product topology of  $\widehat{\mathbb{Z}}$ . Indeed, as noticed in [8, Remark 6.5], we clearly have

$$\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) = \text{Int}_{\mathbb{Q}}\left(\prod_{p \in \mathbb{P}} \pi_p(\underline{E}), \widehat{\mathbb{Z}}\right).$$

Now, we consider the  $\mathbb{Q}$ -Banach space  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$  and the subring  $\mathbb{Q}[x]$  of polynomial functions. Similarly to the first part of the proof of Theorem 3.2, we can see that, if  $\psi \in \mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$  is in the topological closure of  $\mathbb{Q}[x]$ , then the component  $\psi_p$  is a function of only one variable, namely  $\alpha_p$ , and hence, that  $\psi$  belongs to the product  $\prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Q}_p)$ . In fact,  $\psi$  belongs to the intersection

$$\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q})) \cap \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Q}_p)$$

which could be considered as the ‘restricted product’ of the Banach spaces  $\mathcal{C}(E_p, \mathbb{Q}_p)$  with respect to the subrings  $\mathcal{C}(E_p, \mathbb{Z}_p)$ . Now we can state an adelic analogue of Dieudonné’s result (Theorem 1.1):

**Theorem 3.4.** *Let  $\underline{E} = \prod_{p \in \mathbb{P}} E_p$  be a compact subset of  $\mathcal{A}_f(\mathbb{Q})$ . The topological closure of the ring of polynomial functions  $\mathbb{Q}[x]$  in  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$  is the restricted product of the rings  $\mathcal{C}(E_p, \mathbb{Q}_p)$  with respect to the subrings  $\mathcal{C}(E_p, \mathbb{Z}_p)$ , that is:*

$$\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q})) \cap \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Q}_p)$$

*Proof.* Let  $\underline{Q} = \prod_{p \in \mathbb{P}} O_p$  be any basic open neighborhood of 0 in  $\mathcal{A}_f(\mathbb{Q})$ . By Remark 2.2(4), there exist  $d$  and  $d_1 \in \mathbb{N}^*$  such the function  $\varphi$  defined by  $\varphi(\underline{x}) = d_1 \psi(\frac{1}{d} \underline{x})$  belongs to  $\mathcal{C}(d\underline{E}, \widehat{\mathbb{Z}})$  where  $d\underline{E} \subseteq \widehat{\mathbb{Z}}$ . Since  $d_1 \underline{Q} = \prod_p d_1 O_p$  contains a basic open neighborhood of 0 in  $\widehat{\mathbb{Z}}$ , Theorem 3.2 shows that there exists  $f \in \mathbb{Q}[x]$  such that  $(\varphi(\underline{\alpha}) - f(\underline{\alpha})) \in d_1 \underline{Q}$  for every  $\underline{\alpha} \in d\underline{E}$ . Let  $g \in \mathbb{Q}[x]$  defined by  $g(x) = \frac{1}{d_1} f(dx)$ . Then,  $\psi(\underline{\beta}) - g(\underline{\beta}) = \frac{1}{d_1} (\varphi(d\underline{\beta}) - f(d\underline{\beta})) \in \underline{Q}$  for every  $\underline{\beta} \in \underline{E}$ .  $\square$

#### 4. BASES OF THE $\mathbb{Z}$ -MODULE $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$

In order to obtain an analogue of the bases of Theorems 1.3 and 1.4, if there is some, for the  $\mathbb{Q}$ -Banach space  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q})) \cap \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Q}_p)$ , we are looking now for bases of the  $\mathbb{Z}$ -module  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ .

For every compact subset  $\underline{E}$  of  $\widehat{\mathbb{Z}}$ ,  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  is a  $\mathbb{Z}$ -module. Does this  $\mathbb{Z}$ -module admits a basis? As noticed in Remark 3.3.2, we may assume for simplicity that the compact subset  $\underline{E}$  is of the form  $\underline{E} = \prod_p E_p$ , and hence, that (see also [8, (6.1)]):

$$(4) \quad \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) = \bigcap_{p \in \mathbb{P}} \text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p),$$

where  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p) = \{f \in \mathbb{Q}[x] \mid f(E_p) \subseteq \mathbb{Z}_p\}$ , for each  $p \in \mathbb{P}$ . Then, as in the classical case of integer-valued polynomials in number fields, we consider the  $n$ -th characteristic  $\mathbb{Z}$ -module  $\mathfrak{I}_{n, \mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  of  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  which is the set formed by

the leading coefficients (denoted by "lc") of the polynomials of degree  $\leq n$  in  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ :

$$\mathfrak{I}_n(\underline{E}) = \mathfrak{I}_{n, \mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) = \{\text{lc}(f) \mid f \in \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}), \deg(f) \leq n\}.$$

Clearly,

$$\mathbb{Z} = \mathfrak{I}_0(\underline{E}) \subseteq \dots \subseteq \mathfrak{I}_n(\underline{E}) \subseteq \mathfrak{I}_{n+1}(\underline{E}) \subseteq \dots \subseteq \mathbb{Q}$$

Recall that a sequence of polynomials is said to be a *regular basis* in Pólya's sense [12] if this is a basis with exactly one polynomial of each degree. In our setting, [6, Proposition II.1.4] says:

**Lemma 4.1.** *A sequence  $\{f_n\}_{n \geq 0}$  of elements of the  $\mathbb{Z}$ -module  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  is a basis such that  $\deg(f_n) = n$  (and hence a regular basis), if and only if, for each  $n$ , the leading coefficient of  $f_n$  generates the  $\mathbb{Z}$ -module  $\mathfrak{I}_n(\underline{E})$ . In particular,  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  admits a regular basis if and only if all the  $\mathfrak{I}_n(\underline{E})$ 's are (principal) fractional ideals of  $\mathbb{Z}$ .*

Note that the  $\mathbb{Z}$ -module  $\mathfrak{I}_n(\underline{E})$  is a principal fractional ideal of  $\mathbb{Z}$  if and only if it is a finitely generated  $\mathbb{Z}$ -module.

Now, our next task is to study the characteristic  $\mathbb{Z}$ -modules  $\mathfrak{I}_n(\underline{E})$ . Recall that the characteristic  $\mathbb{Z}$ -modules of  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$  are:

$$\mathfrak{I}_n(E_p) = \mathfrak{I}_{n, \mathbb{Q}}(E_p, \mathbb{Z}_p) = \{\text{lc}(f) \mid f \in \text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p), \deg(f) \leq n\}.$$

Note, that, for simplicity, in this section, we write  $\mathfrak{I}_n(\underline{E})$  instead of  $\mathfrak{I}_{n, \mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  and  $\mathfrak{I}_n(E_p)$  instead of  $\mathfrak{I}_{n, \mathbb{Q}}(E_p, \mathbb{Z}_p)$ , but we have to take care that these subsets depend on the field which contains the coefficients of the polynomials. First, we consider the localization with respect to the multiplicative subset  $\mathbb{Z} \setminus p\mathbb{Z}$ . Given a  $\mathbb{Z}$ -module  $M$ , we denote by  $M_{(p)}$  the localization of  $M$  at  $\mathbb{Z} \setminus p\mathbb{Z}$ , which is isomorphic to the tensor product  $M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ .

**Proposition 4.2.** *For every compact subset  $\underline{E} = \prod_{q \in \mathbb{P}} E_q$  and every  $p \in \mathbb{P}$ :*

$$(5) \quad \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p).$$

*In particular,*

$$(6) \quad \mathfrak{I}_n(\underline{E}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \mathfrak{I}_n(E_p),$$

*and hence,*

$$(7) \quad \mathfrak{I}_n(\underline{E}) = \bigcap_{p \in \mathbb{P}} \mathfrak{I}_n(E_p).$$

*Proof.* The isomorphism in (5) is proved once we show the following equality:

$$(8) \quad (\mathbb{Z} \setminus p\mathbb{Z})^{-1} \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) = \text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p),$$

since  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$  is canonically isomorphic to  $(\mathbb{Z} \setminus p\mathbb{Z})^{-1} \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ . Equality (8) is essentially already contained in [8], but for the sake of the reader we give here a self-contained argument.

Clearly the containment  $(\mathbb{Z} \setminus p\mathbb{Z})^{-1} \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) \subseteq \text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$  holds because  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  and  $\mathbb{Z}_{(p)}$  are contained in  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$  (see also (4)). We show now that the other containment holds.

Consider any  $f(x) \in \text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$  and let  $d$  be a positive integer such that  $df(x) \in \mathbb{Z}[x]$ . Write  $d = p^s t$  where  $p \nmid t$  and consider  $g(x) = tf(x)$ . Then,  $g(x) \in \mathbb{Z}_{(q)}[x]$  for all  $q \neq p$ , and hence,  $g(x) \in \text{Int}_{\mathbb{Q}}(E_q, \mathbb{Z}_q)$  for all  $q \neq p$ . Also,



$g(x)$  is in  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$ , because  $t$  is invertible in  $\mathbb{Z}_p$ . Finally,  $g(x) \in \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ ,  $t \in \mathbb{Z} \setminus p\mathbb{Z}$ , and  $f(x) = \frac{1}{t}g(x) \in (\mathbb{Z} \setminus p\mathbb{Z})^{-1} \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ . Equality (8) is proved. The isomorphism (6) and the equality (7) are easy consequences.  $\square$

If  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  admits a regular basis, then all the  $\mathfrak{I}_n(\underline{E})$ 's are fractional ideals of  $\mathbb{Z}$ . In particular, for every  $p \in \mathbb{P}$ , all the  $\mathfrak{I}_n(E_p)$  are fractional ideals of  $\mathbb{Z}$ , which is equivalent to the fact that the  $E_p$ 's are infinite since, for  $n$  fixed,  $\mathfrak{I}_n(E_p)$  is a fractional ideal if and only if  $\#E_p > n$  (cf. [6, Exercise II.3]). The following example shows that this condition is not sufficient.

*Example 4.3.* If  $\underline{E} = \prod_{p \in \mathbb{P}} p\mathbb{Z}$ , then the polynomials  $\frac{1}{p}X$  ( $p \in \mathbb{P}$ ) are in  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ . Therefore the  $\mathbb{Z}$ -module  $\mathfrak{I}_1(\underline{E})$ , which contains (in fact, is equal to) the non-finitely generated  $\mathbb{Z}$ -module  $\sum_{p \in \mathbb{P}} \frac{1}{p}\mathbb{Z}$ , is not a fractional ideal of  $\mathbb{Z}$ , and  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  does not admit a regular basis as a  $\mathbb{Z}$ -module.

**Proposition 4.4.** *Assume that, for each  $p \in \mathbb{P}$ ,  $E_p$  is infinite and let  $\mathfrak{I}_n(E_p) = p^{-n_p}\mathbb{Z}_{(p)}$  where  $n_p \geq 0$ . Then,*

$$(9) \quad \mathfrak{I}_n(\underline{E}) = \sum_{p \in \mathbb{P}} \frac{1}{p^{n_p}} \mathbb{Z}.$$

*In particular,  $\mathfrak{I}_n(\underline{E})$  is a fractional ideal of  $\mathbb{Z}$  if and only if  $\mathfrak{I}_n(E_p) = \mathbb{Z}_{(p)}$  for all but finitely many  $p$ 's. If such a condition holds, then*

$$\mathfrak{I}_n(\underline{E}) = \frac{1}{\prod_{p \in \mathbb{P}} p^{n_p}} \mathbb{Z}.$$

Note that (9) is equivalent to the following equalities:

$$\forall p \in \mathbb{P}, \quad v_p(\mathfrak{I}_n(\underline{E})) = v_p(\mathfrak{I}_n(E_p))$$

where  $v_p(\mathfrak{I}_n(\underline{E})) = \inf_{a \in \mathfrak{I}_n(\underline{E})} v_p(a)$  even if  $\mathfrak{I}_n(\underline{E})$  is not a fractional ideal of  $\mathbb{Z}$ .

*Proof.* We just have to prove equality (9). By hypothesis, there exists a polynomial  $f(x)$  in  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$  of degree  $n$  and leading coefficient  $\frac{1}{p^{n_p}}$ , in fact, such that  $p^{n_p}f(x) \in \mathbb{Z}_{(p)}[x]$  (see [6, Proposition II.1.7]). Now, let  $d$  be an integer coprime to  $p$  such that  $dp^{n_p}f(x) \in \mathbb{Z}[x]$ . Clearly,  $df \in \text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$  and, for each  $q \neq p$ ,  $df \in \mathbb{Z}_{(q)}[x] \subseteq \text{Int}_{\mathbb{Q}}(E_q, \mathbb{Z}_q)$ . Thus,  $df \in \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ . Let  $u$  and  $v$  be two integers such that  $ud + vp^{n_p} = 1$ , then  $\frac{1}{p^{n_p}}$  is the leading coefficient of the polynomial  $udf(x) + vx^n$  which belongs to  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ , that is,  $\frac{1}{p^{n_p}} \in \mathfrak{I}_n(\underline{E})$ . Finally, the  $\mathbb{Z}$ -module  $M = \sum_{p \in \mathbb{P}} \frac{1}{p^{n_p}} \mathbb{Z}$  is contained in  $\mathfrak{I}_n(\underline{E})$ . Conversely, for each  $p \in \mathbb{P}$ , we have the equalities:

$$M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong M_{(p)} = \frac{1}{p^{n_p}} \mathbb{Z}_{(p)} = \mathfrak{I}_n(E_p) \cong \mathfrak{I}_n(\underline{E}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)},$$

which implies  $M = \mathfrak{I}_n(\underline{E})$ .  $\square$

**Corollary 4.5.** *For every compact subset  $\underline{E} = \prod_{p \in \mathbb{P}} E_p$  and every integer  $n$ ,  $\mathfrak{I}_n(\underline{E})$  is a fractional ideal of  $\mathbb{Z}$  if and only if both following conditions hold:*

- (1)  $\#E_p > n$  for all  $p \in \mathbb{P}$ ,
- (2)  $\#(E_p \pmod{p}) > n$  for all but finitely many  $p \in \mathbb{P}$ .

*Proof.* It is enough to show that  $\mathfrak{I}_n(E_p) = \mathbb{Z}_{(p)}$  if and only if  $\#(E_p \pmod{p}) > n$ . Let  $\#(E_p \pmod{p}) = k$  and let  $\alpha_0, \dots, \alpha_{k-1}$  be elements of  $E_p$  non-congruent modulo  $p$ . Assume that  $k \leq n$ , if  $a_0, \dots, a_{k-1}$  are elements of  $\mathbb{Z}$  such that  $a_i \equiv \alpha_i$



(mod  $p$ ), then  $f(x) = \frac{1}{p}x^{n-k}(x-a_0)\dots(x-a_{k-1}) \in \text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$  and  $\frac{1}{p} \in \mathfrak{I}_n(E_p)$ . Assume now that  $k \geq n+1$ , then whatever  $1 \leq l \leq n+1$ ,  $v_p(\prod_{h=0}^{l-1}(\alpha_l - \alpha_h)) = 0$ , which implies that  $\alpha_0, \dots, \alpha_{n+1}$  is the beginning of a  $p$ -ordering of  $E_p$  and, in particular, the fact that  $v_p(\prod_{h=0}^n(\alpha_{n+1} - \alpha_h)) = 0$  means that  $\mathfrak{I}_n(E_p) = \mathbb{Z}_{(p)}$  (about  $p$ -orderings and links with characteristic ideals see [2]).  $\square$

Lemma 4.1 with the previous corollary implies that:

**Corollary 4.6.** *For every compact subset  $\underline{E} = \prod_{p \in \mathbb{P}} E_p$  of  $\widehat{\mathbb{Z}}$ , the  $\mathbb{Z}$ -module  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  admits a regular basis if and only if, both following conditions hold:*

- (1)  $\#E_p$  is infinite for all  $p \in \mathbb{P}$ ,
- (2) for each  $n$ ,  $\#(E_p \pmod{p}) > n$  for all but finitely many  $p \in \mathbb{P}$ .

*Remark 4.7.* We recall that the family of overrings of  $\text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$  contained in  $\mathbb{Q}[x]$  is formed exactly by the rings  $\text{Int}(\underline{E}, \widehat{\mathbb{Z}})$ , as  $\underline{E}$  ranges through the subsets of  $\widehat{\mathbb{Z}}$  of the form  $\prod_p E_p$ , where for each prime  $p$ ,  $E_p$  is a closed subset of  $\mathbb{Z}_p$  (see [8, Theorem 6.4]). Among these rings we can find the subfamily of integer-valued polynomials rings over an infinite subset  $E$  of  $\mathbb{Z}$ , that is,  $\text{Int}(E, \mathbb{Z}) = \{f \in \mathbb{Q}[x] \mid f(E) \subseteq \mathbb{Z}\}$ . As already recalled just after Proposition 4.2, each ring of the last subfamily has a regular basis (if  $E$  is an infinite subset of  $\mathbb{Z}$ , each of the characteristic ideals of  $\text{Int}(E, \mathbb{Z})$  is a finitely generated  $\mathbb{Z}$ -module [6, Corollary II.1.6]). We show in Example 4.8 below that there are rings  $\text{Int}(\underline{E}, \widehat{\mathbb{Z}})$  which have a regular basis but are not of the form  $\text{Int}(E, \mathbb{Z})$  for any infinite subset  $E$  of  $\mathbb{Z}$ .

*Example 4.8.* For each  $p \in \mathbb{P}$ , choose  $\alpha_p \in \mathbb{Z}_p \setminus \mathbb{Z}_{(p)}$  and let  $E_p = \{\alpha_p + j + p^n \mid 0 \leq j \leq p-1, n \geq 0\}$ . The set  $E_p$  is  $p$ -adically closed since it is the union of  $p$  convergent sequences. Moreover,  $\#(E_p \pmod{p}) = p$ , and hence,  $\text{Int}_{\mathbb{Q}}(\prod_p E_p, \widehat{\mathbb{Z}})$  has regular bases although it is not of the form  $\text{Int}(E, \mathbb{Z})$  for some  $E \subseteq \mathbb{Z}$ , by [8, Corollary 6.9].

If the  $\mathbb{Z}$ -module  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  admits a regular basis  $\{f_n\}_{n \geq 0}$ , then clearly, such a basis is also a regular basis of the  $\mathbb{Z}_{(p)}$ -module  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$  for every  $p \in \mathbb{P}$ , by (5). Conversely, from regular bases  $\{f_{n,p}\}_{n \geq 0}$  of  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$  for every  $p \in \mathbb{P}$ , one may construct a regular basis of  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ . We show how we can do this. But, first, we recall how one can construct a basis of the  $\mathbb{Z}_p$ -module  $\text{Int}(E_p, \mathbb{Z}_p) = \text{Int}_{\mathbb{Q}_p}(E_p, \mathbb{Z}_p) = \{f \in \mathbb{Q}_p[x] \mid f(E_p) \subseteq \mathbb{Z}_p\}$  by means of the notion of Bhargava's  $p$ -ordering. From this basis we will deduce a basis for the  $\mathbb{Z}_{(p)}$ -module  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p) = \text{Int}(E_p, \mathbb{Z}_p) \cap \mathbb{Q}[x]$  using the fact that

$$(10) \quad \text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \text{Int}(E_p, \mathbb{Z}_p)$$

Recall that a  $p$ -ordering of a subset  $E_p$  of  $\mathbb{Z}_p$  is a sequence  $\{a_n\}_{n \geq 0}$  of elements of  $E_p$  such that:

$$\forall n \geq 1, \quad v_p\left(\prod_{k=0}^{n-1} (a_n - a_k)\right) = \min_{x \in E_p} v_p\left(\prod_{k=0}^{n-1} (x - a_k)\right).$$

Such a sequence always exists, and its elements are distinct if and only if  $E_p$  is infinite. Then, it is quite obvious that the polynomials

$$f_{0,p} = 1 \quad \text{and, for } n \geq 1, \quad f_{n,p}(x) = \prod_{k=0}^{n-1} \frac{x - a_k}{a_n - a_k}$$

form a regular basis of the  $\mathbb{Z}_p$ -module  $\text{Int}(E_p, \mathbb{Z}_p)$  (cf. Bhargava [3]).

For each  $n \in \mathbb{N}$ , we set

$$g_{n,p}(x) = \prod_{k=0}^{n-1} (x - a_k), \quad w_p(n) = v_p\left(\prod_{k=0}^{n-1} (a_n - a_k)\right).$$

It is clear that  $g_{n,p}(E_p) \subseteq p^{w_p(n)}\mathbb{Z}_p$ . Let  $h_{n,p} \in \mathbb{Z}[x]$  be of degree  $n$  such that  $h_{n,p} \equiv g_{n,p} \pmod{p^{w_p(n)}}$ . It is immediate to see that  $\frac{h_{n,p}(x)}{p^{w_p(n)}} \in \text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$ .

Therefore the polynomials  $\frac{h_{n,p}(x)}{p^{w_p(n)}}$ ,  $n \in \mathbb{N}$ , form a basis of  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$ .

### Construction of a $\mathbb{Z}$ -basis of $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ .

Suppose that for each  $p \in \mathbb{P}$ ,  $E_p$  is an infinite subset of  $\mathbb{Z}_p$  and let  $\{f_{n,p}\}_{n \in \mathbb{N}}$  be a regular basis of  $\text{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p)$ . For a fixed  $n$ , let  $P_n$  be the set of primes  $p$  such that  $\#(E \pmod{p}) \leq n$ . By Corollary 4.6,  $P_n$  is finite. By the Chinese remainder theorem (cf. Remark 3.3.1), there exists  $f_n \in \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  such that  $f_n \equiv f_{n,p} \pmod{p}$  for every  $p \in P_n$  and  $f_n \in \mathbb{Z}_{(q)}[x]$  for every  $q \in \mathbb{P} \setminus P_n$ . Let  $c_n$  and  $c_{n,p}$  denote the leading coefficients of  $f_n$  and  $f_{n,p}$ , respectively. Then, for  $p \in P_n$ ,  $v_p(c_{n,p}) < 0$  and  $v_p(c_n - c_{n,p}) > 0$  implies  $v_p(c_n) = v_p(c_{n,p}) = v_p(\mathfrak{I}_n(E_p)) = v_p(\mathfrak{I}_n(\underline{E}))$ . Write  $c_n = \frac{a}{b}$  where  $(a, b) = 1$ . Consequently, for every  $p \in P_n$ ,  $v_p(c_n) = -v_p(b) = v_p(\mathfrak{I}_n(\underline{E}))$  and, for every  $q \in \mathbb{P} \setminus P_n$ ,  $v_q(c_n) = v_q(a) \geq 0$ . Let  $u, v \in \mathbb{Z}$  be such that  $ua + vb = 1$  and consider the polynomial  $g_n = uf_n + vx^n \in \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  with  $\deg(g_n) = n$  and leading coefficient  $\frac{1}{b}$ . Finally, by Lemma 4.1, since  $v_p(\frac{1}{b}) = v_p(\mathfrak{I}_n(\underline{E}))$  for all  $p \in \mathbb{P}$ , the  $g_n$ 's form a regular basis of  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ .

The next remark explains why finally the ring  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  is not enough for the purpose of writing any element of the  $\mathbb{Q}$ -Banach space  $\prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Q}_p)$  as a sum of a series in the elements of a regular basis of  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  with integer coefficients.

*Remark 4.9.* Assume that the  $\mathbb{Z}$ -module  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  admits a regular basis  $\{f_n\}_{n \geq 0}$ . Although, by Theorem 3.2, the  $\mathbb{Z}$ -linear combinations of the  $f_n$ 's may uniformly approximate every element  $\underline{\varphi} = (\varphi_p)_{p \in \mathbb{P}}$  of the ring  $\prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p)$ , these  $\mathbb{Z}$ -linear combinations  $\sum_{n=0}^d c_n f_n$  are not enough to allow to write such a  $\underline{\varphi}$  as the sum of a series in the  $f_n$ 's with coefficients in  $\mathbb{Z}$ . The reason is that the coefficients  $c_n$ 's of the  $f_n$ 's strongly depend on the neighborhood  $O$  chosen for the approximation of  $\underline{\varphi}$ : it follows from Theorem 1.4 that, for a decreasing sequence of neighborhoods of 0 of the form  $O_l = p_0^l \mathbb{Z}_{p_0} \times \prod_{p \neq p_0} \mathbb{Z}_p$ , the coefficients  $c_n$  are uniquely determined by the component  $\varphi_{p_0}$  of  $\underline{\varphi}$ . And, generally, they are different for another component  $\varphi_p$  since the components of  $\underline{\varphi}$  may be chosen independently of each other. Thus, we will have to consider coefficients in  $\mathcal{A}_f(\mathbb{Q})$  instead of  $\mathbb{Q}$  and this leads to the next section.

## 5. POLYNOMIALS WITH ADELIC COEFFICIENTS

Let us consider now polynomials  $\underline{g}(x)$  with coefficients in  $\mathcal{A}_f(\mathbb{Q})$ :

$$\underline{g}(x) = \sum_{k=0}^n \underline{\gamma}_k x^k, \quad \text{where } \underline{\gamma}_k = (\gamma_{k,p})_{p \in \mathbb{P}} \in \mathcal{A}_f(\mathbb{Q}).$$

For such a polynomial  $\underline{g}(x)$ , we introduce its components:

$$\underline{g} = (g_p)_{p \in \mathbb{P}}, \text{ where } g_p(x) = \sum_{k=0}^n \gamma_{k,p} x^k \text{ with } \gamma_{k,p} \in \mathbb{Q}_p$$

corresponding to the containment  $\mathcal{A}_f(\mathbb{Q})[x] \subset \prod_{p \in \mathbb{P}} \mathbb{Q}_p[x]$ . Note that the last containment is strict, even if we consider the restricted product of the  $\mathbb{Q}_p[x]$ 's with respect to the  $\mathbb{Z}_p[x]$ 's because of the degrees of the  $g_p$ 's which are bounded: for every  $\underline{g} \in \mathcal{A}_f(\mathbb{Q})[x]$ ,  $\deg(g_p) \leq \deg(\underline{g})$  for all  $p$ .

Similarly to the case of  $\mathbb{Q}[x]$ , for  $\underline{g} \in \mathcal{A}_f(\mathbb{Q})[x]$  and  $\underline{\alpha} \in \mathcal{A}_f(\mathbb{Q})$ , we may consider the value of  $\underline{g}(x)$  at  $\underline{\alpha}$ :

$$\underline{g}(\underline{\alpha}) = \sum_{k=0}^n \underline{\gamma}_k \underline{\alpha}^k = \left( \sum_{k=0}^n \gamma_{k,p} \alpha_p^k \right)_{p \in \mathbb{P}} = (g_p(\alpha_p))_{p \in \mathbb{P}}.$$

Thus, every  $\underline{g} \in \mathcal{A}_f(\mathbb{Q})[x]$  may be considered as a function from  $\mathcal{A}_f(\mathbb{Q})$  into itself:

$$\forall \underline{g} = (g_p)_{p \in \mathbb{P}} \in \mathcal{A}_f(\mathbb{Q})[x], \quad \mathcal{A}_f(\mathbb{Q}) \ni \underline{\alpha} = (\alpha_p) \mapsto \underline{g}(\underline{\alpha}) = (g_p(\alpha_p)) \in \mathcal{A}_f(\mathbb{Q})$$

and, analogously to (3), we have the following containment:

$$\mathcal{A}_f(\mathbb{Q})[x] \subset \mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q})),$$

whatever  $\underline{E} \subseteq \mathcal{A}_f(\mathbb{Q})$ . Moreover, as for  $\mathbb{Q}[x]$ , if the compact subset  $\underline{E}$  is of the form  $\prod_{p \in \mathbb{P}} E_p$ , we also have the following containment:

$$\mathcal{A}_f(\mathbb{Q})[x] \subset \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Q}_p).$$

Indeed, let  $\underline{g} = (g_p)_{p \in \mathbb{P}} \in \mathcal{A}_f(\mathbb{Q})[x]$  and fix some  $p \in \mathbb{P}$ . The fact that  $(\underline{g}(\underline{\alpha}))_p = g_p(\alpha_p)$  means that the value of the  $p$ -component  $g_p(x)$  of  $\underline{g}(x)$  at  $\underline{\alpha}$  depends only on  $\alpha_p$ , that is,  $g_p \in \mathcal{C}(E_p, \mathbb{Q}_p)$ . Putting together all these containments, we obtain:

$$(11) \quad \mathbb{Q}[x] \subset \mathcal{A}_f(\mathbb{Q})[x] \subset \mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q})) \cap \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Q}_p) \subset \mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q})),$$

which shows in particular that the topological closure of  $\mathcal{A}_f(\mathbb{Q})[x]$  in  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q}))$  with respect to the uniform convergence topology is equal to topological closure of  $\mathbb{Q}[x]$ , namely,  $\mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q})) \cap \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Q}_p)$  (Theorem 3.4).

Considering now polynomial functions with values in  $\widehat{\mathbb{Z}}$  instead of  $\mathcal{A}_f(\mathbb{Q})$  (note that  $\mathcal{A}_f(\mathbb{Q})$  is equal to  $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ , the localization of  $\widehat{\mathbb{Z}}$  at  $\mathbb{Z} \setminus \{0\}$ ), we are led to introduce the following polynomial ring of integer-valued polynomials on  $\underline{E} = \prod_p E_p$ :

$$\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}}) := \{\underline{g} \in \mathcal{A}_f(\mathbb{Q})[x] \mid \underline{g}(\underline{E}) \subseteq \widehat{\mathbb{Z}}\}.$$

Clearly, by definition:

$$\forall \underline{g} = (g_p)_p \in \mathcal{A}_f(\mathbb{Q})[x], \quad \underline{g} \in \text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}}) \Leftrightarrow g_p(E_p) \subseteq \mathbb{Z}_p, \quad \forall p \in \mathbb{P}.$$

Consequently, for every  $\underline{g} = (g_p)_p \in \mathcal{A}_f(\mathbb{Q})[x]$ , if  $\underline{g} \in \text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$  then, for all but finitely many  $p \in \mathbb{P}$ ,  $g_p \in \mathbb{Z}_p[x]$  and, for the other  $p$ ,  $g_p \in \text{Int}_{\mathbb{Q}_p}(E_p, \mathbb{Z}_p)$ , in other words,  $\underline{g}$  belongs to the restricted product of the rings  $\text{Int}_{\mathbb{Q}_p}(E_p, \mathbb{Z}_p)$  with respect to the subrings  $\mathbb{Z}_p[x]$ . Note that

$$\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}}) \cap \mathbb{Q}[x] = \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}).$$

By intersection with  $\mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$ , containments (11) lead to:

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) \subset \mathrm{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}}) \subset \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p) \subset \mathcal{C}(\underline{E}, \widehat{\mathbb{Z}}).$$

Once again, in order to study  $\mathrm{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ , we introduce its characteristic modules:

$$\mathfrak{I}_{n, \mathcal{A}_f}(\underline{E}) = \mathfrak{I}_{n, \mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}}) := \{\mathrm{lc}(\underline{g}) \mid \underline{g} \in \mathrm{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}}), \deg(\underline{g}) \leq n\}.$$

This is a sub- $\mathbb{Z}$ -module of  $\mathcal{A}_f(\mathbb{Q})$ . It is easy to see that

$$\pi_p(\mathfrak{I}_{n, \mathcal{A}_f}(\underline{E})) = \mathfrak{I}_{n, \mathbb{Q}_p}(E_p)$$

where

$$\mathfrak{I}_{n, \mathbb{Q}_p}(E_p) := \{\mathrm{lc}(f) \mid f \in \mathrm{Int}_{\mathbb{Q}_p}(E_p, \mathbb{Z}_p), \deg(f) \leq n\}.$$

Indeed, if  $\underline{g} \in \mathrm{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$  then, for every  $p \in \mathbb{P}$ ,  $g_p \in \mathrm{Int}_{\mathbb{Q}_p}(E_p, \mathbb{Z}_p)$ , and hence,  $\pi_p(\mathfrak{I}_{n, \mathcal{A}_f}(\underline{E})) \subseteq \mathfrak{I}_{n, \mathbb{Q}_p}(E_p)$ . Conversely, if  $c \in \mathfrak{I}_{n, \mathbb{Q}_p}(E_p)$ , there exists  $f \in \mathrm{Int}_{\mathbb{Q}_p}(E_p, \mathbb{Z}_p)$  with leading coefficient  $c$ . Let  $\underline{g} \in \mathrm{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$  be such that  $g_p = f$  and  $g_q = 0$  for all  $q \neq p$ , then  $\pi_p(\mathrm{lc}(\underline{g})) = c$ .

Recall that:

$$\mathfrak{I}_n(E_p) = \{\mathrm{lc}(f) \mid f \in \mathrm{Int}_{\mathbb{Q}}(E_p, \mathbb{Z}_p), \deg(f) \leq n\},$$

$$\mathfrak{I}_n(\underline{E}) = \{\mathrm{lc}(\underline{g}) \mid \underline{g} \in \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}), \deg(\underline{g}) \leq n\}.$$

We saw in Proposition 4.2 that

$$\mathfrak{I}_n(\underline{E}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \mathfrak{I}_n(E_p)$$

and it is straightforward that (see (10))

$$\mathfrak{I}_n(E_p) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathfrak{I}_{n, \mathbb{Q}_p}(E_p).$$

The next proposition gives the link between  $\mathrm{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$  and  $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ :

**Theorem 5.1.** *For every  $\underline{E} = \prod_{p \in \mathbb{P}} E_p \subseteq \mathcal{A}_f(\mathbb{Q})$ , we have*

$$\mathrm{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}}) \cong \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}.$$

In particular,

$$\mathfrak{I}_{n, \mathcal{A}_f}(\underline{E}) \cong \mathfrak{I}_n(\underline{E}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}, \quad \forall n \geq 0.$$

and if  $\{f_n\}_{n \in \mathbb{N}}$  is a regular basis of the  $\mathbb{Z}$ -module  $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ , then  $\{f_n\}_{n \in \mathbb{N}}$  is a regular basis of the  $\widehat{\mathbb{Z}}$ -module  $\mathrm{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ .

*Proof.* Let  $\eta : \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \cong \mathcal{A}_f(\mathbb{Q})$  be the canonical isomorphism of  $\widehat{\mathbb{Z}}$ -modules characterized by  $\eta(r \otimes \underline{1}) = j(r) = r\underline{1}$  where  $j$  denotes the embedding  $\mathbb{Q} \rightarrow \mathcal{A}_f(\mathbb{Q})$  defined by (1). Then,  $\eta$  induces another canonical isomorphism of  $\widehat{\mathbb{Z}}$ -modules  $\tilde{\eta} : \mathbb{Q}[x] \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \cong \mathcal{A}_f(\mathbb{Q})[x]$ , itself characterized by:

$$(12) \quad \forall g(x) = \sum_{n=0}^d c_n x^n \in \mathbb{Q}[x], \quad \tilde{\eta}(g(x) \otimes \underline{1}) = \sum_{n=0}^d j(c_n) x^n = \underline{1}g(x).$$

With these identifications, we may consider that we have the following containment  $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \subseteq \mathcal{A}_f(\mathbb{Q})[x]$  since  $\widehat{\mathbb{Z}}$  is a faithfully flat  $\mathbb{Z}$ -module (see for instance [5, Chap. I, §. 3, 1. Exemples]). In fact, by formulas (2) and (12), it is clear that the evaluation of any polynomial in  $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$  at any element of  $\underline{E}$  is in  $\widehat{\mathbb{Z}}$ ,

so it follows that  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$  may be considered a subring of  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ . Conversely, let

$$\underline{g}(x) = (g_p(x)) = \sum_{n=0}^d \gamma_{n,p} x^n$$

be any element of  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ . We have to show that  $\underline{g}(x)$  can be written as a finite linear combination of elements of  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  with coefficients in  $\widehat{\mathbb{Z}}$ . There is a finite set  $P_0 = \{p_i \mid i \in I\}$  of primes such that  $g_p \in \mathbb{Z}_p[x]$  for  $p \notin P_0$ , and we know that, for  $p \in P_0$ ,  $g_p \in \text{Int}_{\mathbb{Q}_p}(E_p, \mathbb{Z}_p)$ . For  $0 \leq n \leq d$ , we define  $\underline{\delta}_n \in \widehat{\mathbb{Z}}$  by  $\delta_{n,p} = \gamma_{n,p}$  for  $p \notin P_0$  and  $\delta_{n,p} = 0$  for  $p \in P_0$ . Since, for each  $p$ ,  $\text{Int}_{\mathbb{Q}_p}(E_p, \mathbb{Z}_p) \cong \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$  (by (5) and (10)), there exists a finite set of polynomials  $\{h_j \mid j \in J\} \subset \text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  such that, for  $p \in P_0$ , we may write  $g_p(x) = \sum_{j \in J} \varepsilon_{j,p} h_j(x)$  where  $\varepsilon_{j,p} \in \mathbb{Z}_p$ . Now, for  $0 \leq i \leq s$ , let  $\underline{\nu}_i = (\nu_{i,p}) \in \widehat{\mathbb{Z}}$  with  $\nu_{i,p_i} = 1$  and  $\nu_{i,p} = 0$  for  $p \neq p_i$ . Finally, we have:

$$\underline{g}(x) = \sum_{n=0}^d \underline{\delta}_n x^n + \sum_{j \in J} \left( \sum_{i \in I} \underline{\nu}_i \varepsilon_{j,p_i} \right) h_j(x),$$

which shows that  $\underline{g}(x)$  is a finite linear combination of elements of  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  with coefficients in  $\widehat{\mathbb{Z}}$ .  $\square$

In this adelic framework, Lemma 4.1 becomes:

**Lemma 5.2.** *A sequence  $\{f_n\}_{n \geq 0}$  of elements of the  $\widehat{\mathbb{Z}}$ -module  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$  is a regular basis if and only if, for each  $n$ , the leading coefficient of  $f_n$  generates the  $\widehat{\mathbb{Z}}$ -module  $\mathfrak{I}_{n, \mathcal{A}_f}(\underline{E})$ . In particular,  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$  admits a regular basis if and only if the  $\mathfrak{I}_{n, \mathcal{A}_f}(\underline{E})$ 's are cyclic  $\widehat{\mathbb{Z}}$ -modules.*

**Proposition 5.3.** *The  $\widehat{\mathbb{Z}}$ -module  $\mathfrak{I}_{n, \mathcal{A}_f}(\underline{E})$  is cyclic if and only if the  $\mathbb{Z}$ -module  $\mathfrak{I}_n(\underline{E})$  is cyclic.*

*Proof.* It is clear that, if  $\mathfrak{I}_n(\underline{E})$  is cyclic, then  $\mathfrak{I}_{n, \mathcal{A}_f}(\underline{E}) = \mathfrak{I}_n(\underline{E}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$  is also cyclic. Conversely, if  $\mathfrak{I}_{n, \mathcal{A}_f}(\underline{E}) = \underline{\alpha} \widehat{\mathbb{Z}}$  where  $\underline{\alpha} \in \mathcal{A}_f(\mathbb{Q}) = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ , then there exists  $d \in \mathbb{Z} \setminus \{0\}$  such that  $d\underline{\alpha} \in \widehat{\mathbb{Z}}$ , and hence,  $d\mathfrak{I}_n(\underline{E}) \subseteq \mathbb{Z}$ .  $\square$

It follows from Lemma 4.1, Lemma 5.2 and Proposition 5.3 that if the  $\widehat{\mathbb{Z}}$ -module  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$  has a regular basis then the  $\mathbb{Z}$ -module  $\text{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$  has a regular basis.

Recall that, following Corollary 4.6, all the characteristic modules are cyclic if and only if

- (1)  $\#E_p$  is infinite for all  $p \in \mathbb{P}$ ,
- (2) for each  $n \in \mathbb{N}$ ,  $\#(E_p \pmod{p}) > n$  for all but finitely many  $p \in \mathbb{P}$ .

Assuming that these conditions are satisfied, we show now how we can construct regular bases of the  $\widehat{\mathbb{Z}}$ -module  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ . We first extend Bhargava's notion of  $p$ -ordering.

**Definition 5.4.** An *adelic ordering* of  $\underline{E}$  is a sequence  $\{\underline{\alpha}_n\}_{n \geq 0}$  of elements of  $\underline{E}$  such that both following conditions hold:

- (a)  $\forall p \in \mathbb{P}, \forall n \geq 1, v_p \left( \pi_p \left( \prod_{k=0}^{n-1} (\underline{\alpha}_n - \underline{\alpha}_k) \right) \right) = \min_{\underline{y} \in \underline{E}} v_p \left( \pi_p \left( \prod_{k=0}^{n-1} (\underline{y} - \underline{\alpha}_k) \right) \right)$

(b)  $\forall n \geq 1, v_p \left( \pi_p \left( \prod_{k=0}^{n-1} (\underline{\alpha}_n - \underline{\alpha}_k) \right) \right) = 0$  for almost  $p \in \mathbb{P}$ .

The first condition means that, for each  $p \in \mathbb{P}$ , the sequence  $\{\alpha_{n,p}\}$  is a  $p$ -ordering of  $E_p$ , while the second condition means that the product  $\prod_{k=0}^{n-1} (\underline{\alpha}_n - \underline{\alpha}_k)$  is invertible in  $\mathcal{A}_f(\mathbb{Q})$ . Since  $v_p(\pi_p(\prod_{k=0}^{n-1} (\underline{\alpha}_n - \underline{\alpha}_k))) = v_p(\prod_{k=0}^{n-1} (\alpha_{n,p} - \alpha_{k,p}))$  which is the value of the  $p$ -sequence of  $E_p$  at  $n$ , condition b) above is easily seen to be equivalent to condition (2) of Corollary 4.5.

**Proposition 5.5.** *Let  $\{\underline{\alpha}_n\}_{n \geq 0}$  be a sequence of elements of  $\underline{E}$  such that, for each  $p$ , the  $\alpha_{n,p}$ 's are distinct. Consider the associated polynomials:*

$$\underline{g}_0 = 1 \text{ and, for } n \geq 1, \underline{g}_n(x) = \prod_{k=0}^{n-1} \frac{x - \underline{\alpha}_k}{\underline{\alpha}_n - \underline{\alpha}_k}.$$

The following assertions are equivalent:

- (1) The sequence  $\{\underline{\alpha}_n\}_{n \geq 0}$  is an adelic ordering of  $\underline{E}$ .
- (2) The polynomials  $\underline{g}_n$  belong to  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ .
- (3) The polynomials  $\underline{g}_n$  form a regular basis of the  $\widehat{\mathbb{Z}}$ -module  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ .
- (4) For every polynomial  $\underline{g} \in \mathcal{A}_f(\mathbb{Q})[x]$  of degree  $n$ ,

$$\underline{g} \in \text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}}) \Leftrightarrow \underline{g}(\underline{\alpha}_k) \in \widehat{\mathbb{Z}} \text{ for } 0 \leq k \leq n.$$

*Proof.* Let us prove first that (1) $\leftrightarrow$ (2): the coefficients of the polynomials  $\underline{g}_n$  belong a priori to  $\prod_{p \in \mathbb{P}} \mathbb{Q}_p$ , to say that there are in  $\mathcal{A}_f(\mathbb{Q})$  is equivalent to the second condition of Definition 5.4. Moreover,  $\underline{g}_n \in \text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$  is equivalent to: for every  $p \in \mathbb{P}$ ,  $\underline{g}_{n,p}(x) = \prod_{k=0}^{n-1} \frac{x - \alpha_{k,p}}{\alpha_{n,p} - \alpha_{k,p}} \in \text{Int}_{\mathbb{Q}_p}(E_p, \mathbb{Z}_p)$ , which is known to be equivalent to the fact that the sequence  $\{\alpha_{n,p}\}_{n \geq 0}$  is a  $p$ -ordering of  $E_p$  (cf. [2]), which is the first condition of Definition 5.4.

Clearly, (3) $\rightarrow$ (2). Conversely, assume that the  $\underline{g}_k$ 's are in  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ . Because of the degrees, every  $\underline{g} \in \text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$  of degree  $\leq n$  may be written  $\underline{g}(x) = \sum_{k=0}^n \underline{c}_k \underline{g}_k(x)$  with  $\underline{c}_k \in \mathcal{A}_f(\mathbb{Q})$ . Since, for every  $k$ ,  $\underline{g}_k(\underline{\alpha}_h) = \underline{0}$  for  $0 \leq h \leq k-1$  and  $\underline{g}_k(\underline{\alpha}_k) = \underline{1}$ , it is easy to see by induction on  $k$  that  $\underline{g}(\underline{\alpha}_h) \in \widehat{\mathbb{Z}}$  for  $0 \leq h \leq k$  implies that the coefficients  $\underline{c}_0, \dots, \underline{c}_k$  belong to  $\widehat{\mathbb{Z}}$ . Indeed, the coefficients  $\underline{c}_k$  may be computed recursively by:

$$(13) \quad \underline{c}_k = \underline{g}(\underline{\alpha}_k) - \sum_{h=0}^{k-1} \underline{c}_h \underline{g}_h(\underline{\alpha}_k).$$

Consequently, the  $\underline{g}_k$ 's form a  $\mathbb{Z}$ -basis, so that (2) $\rightarrow$ (3).

Finally that (3) $\leftrightarrow$ (4) follows easily from the previous proof of (2) $\rightarrow$ (3).  $\square$

*Example 5.6.* The sequence  $\{\underline{n}\}_{n \geq 0}$  (where  $\underline{n} = (n_p)_{p \in \mathbb{P}}$  and  $n_p = n$  for every  $p$ ) is an adelic ordering of  $\widehat{\mathbb{Z}}$ . Thus, the polynomials  $\binom{x}{n}$  form a regular basis of  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\widehat{\mathbb{Z}})$ : every  $\underline{g} \in \text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\widehat{\mathbb{Z}})$  of degree  $n$  may be uniquely written as  $\sum_{k=0}^n \underline{c}_k \binom{x}{k}$  where the  $\underline{c}_k$ 's are in  $\widehat{\mathbb{Z}}$  and satisfy:

$$\underline{c}_k = \underline{g}(\underline{k}) - \sum_{h=0}^{k-1} \underline{c}_h \binom{k}{h} \in \widehat{\mathbb{Z}}.$$

## 6. EXTENSION OF BHARGAVA-KEDLAYA'S THEOREM

Recall that, in  $\widehat{\mathbb{Z}}$ , a sequence  $\{\underline{\alpha}_n\}_{n \geq 0} = \{(\alpha_{n,p})_{p \in \mathbb{P}}\}_{n \geq 0}$  converges to  $\underline{\alpha} = (\alpha_p)_{p \in \mathbb{P}}$  if and only if, for each  $p \in \mathbb{P}$ , the sequence  $\{\alpha_{n,p}\}_{n \geq 0}$  converges to  $\alpha_p$  in  $\mathbb{Z}_p$ . And, in  $\mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$ , a sequence  $\{\underline{\varphi}_n\}_{n \geq 0} = \{(\varphi_{n,p})_{p \in \mathbb{P}}\}_{n \geq 0} \in \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p)$  converges uniformly to  $\underline{\varphi} = (\varphi_p)_{p \in \mathbb{P}}$  if and only if, for each  $p \in \mathbb{P}$ ,  $\{\varphi_{n,p}\}_{n \geq 0}$  converges uniformly to  $\varphi_p$  in  $\mathcal{C}(E_p, \mathbb{Z}_p)$ .

It follows from the previous section that Mahler's result (Theorem 1.3) extends in the following way:

**Proposition 6.1.** *Every  $\underline{\varphi} \in \prod_{p \in \mathbb{P}} \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$  may be uniquely written as a series in the  $\binom{x}{n}$ 's with coefficients in  $\widehat{\mathbb{Z}}$ :*

$$\underline{\varphi}(x) = \sum_{n=0}^{\infty} \underline{c}_n \binom{x}{n}, \text{ where } \underline{c}_n \in \widehat{\mathbb{Z}}, \lim_{n \rightarrow +\infty} \underline{c}_n = \underline{0}.$$

Moreover,

$$\underline{c}_k = \underline{\varphi}(\underline{k}) - \sum_{h=0}^{k-1} \underline{c}_h \binom{k}{h} \in \widehat{\mathbb{Z}}.$$

More generally and more precisely, we extend Bhargava-Kedlaya's result (Theorem 1.4):

**Theorem 6.2.** *Assume that  $\underline{E} = \prod_{p \in \mathbb{P}} E_p$  is a compact subset of  $\widehat{\mathbb{Z}}$  such that, for each  $p \in \mathbb{P}$ ,  $E_p$  is infinite and, for each  $n \geq 0$ ,  $\#(E \pmod{p}) > n$  for almost all  $p \in \mathbb{P}$ . Then, firstly there exists a sequence  $\{\underline{\alpha}_n\}_{n \geq 0}$  which is an adelic ordering of  $\underline{E}$ , and the corresponding polynomials  $\underline{g}_n(x) = \prod_{k=0}^{n-1} \frac{x - \underline{\alpha}_k}{\underline{\alpha}_n - \underline{\alpha}_k}$  form a basis of  $\widehat{\mathbb{Z}}$ -module  $\text{Int}_{A_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ . Secondly, every  $\underline{\varphi} \in \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Z}_p)$  may be uniquely written as a (uniformly convergent) series in the  $\underline{g}_n$ 's with coefficients in  $\widehat{\mathbb{Z}}$ :*

$$\underline{\varphi}(x) = \sum_{n \geq 0} \underline{c}_n \underline{g}_n(x) \text{ where } \underline{c}_n \in \widehat{\mathbb{Z}} \text{ and } \lim_{n \rightarrow +\infty} \underline{c}_n = \underline{0}.$$

Moreover, the  $\underline{c}_n$ 's satisfy the recursive formula:

$$(14) \quad \underline{c}_n = \underline{\varphi}(\underline{\alpha}_n) - \sum_{k=0}^{n-1} \underline{c}_k \underline{g}_k(\underline{\alpha}_n),$$

and one has:

$$\forall p \in \mathbb{P}, \quad \inf_{n \geq 0} v_p(\pi_p(\underline{c}_n)) = \inf_{\underline{y} \in \underline{E}} v_p(\pi_p(\underline{\varphi}(\underline{y}))).$$

*Proof.* The first assertion follows from Corollary 4.6, Theorem 5.1, Proposition 5.3 and Proposition 5.5. The second assertion is then a straightforward consequence of Theorem 1.4 since  $\underline{\varphi} = (\varphi_p) \in \prod_p \mathcal{C}(E_p, \mathbb{Z}_p)$  and, if the  $\underline{g}_n$ 's form a basis of the  $\widehat{\mathbb{Z}}$ -module  $\text{Int}_{A_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$  then, for every  $p \in \mathbb{P}$ , the  $\pi_p(\underline{g}_n)$ 's form a basis of the  $\mathbb{Z}_p$ -module  $\text{Int}_{\mathbb{Q}_p}(E_p, \mathbb{Z}_p)$ , by definition of adelic ordering. The last assertions are consequences of formula (13), Theorem 1.4 and the fact that since  $\{\underline{\alpha}_n\}_{n \geq 0}$  is an adelic ordering of  $\underline{E}$  then, for each  $p \in \mathbb{P}$ ,  $\{\pi_p(\underline{\alpha}_n)\}_{n \geq 0}$  is a  $p$ -ordering of  $E_p = \pi_p(\underline{E})$ .  $\square$



In fact, we can write the functions  $\varphi$  as sums of series of polynomials even if these polynomials are not associated to an adelic ordering of  $\underline{E}$ . It is enough that they form a basis of the  $\widehat{\mathbb{Z}}$ -module  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ . Moreover, thanks to Remark 2.2, we may consider functions  $\varphi$  defined on a compact subset of  $\mathcal{A}_f(\mathbb{Q})$  and with values in  $\mathcal{A}_f(\mathbb{Q})$ .

**Theorem 6.3.** *Assume that  $\underline{E} = \prod_{p \in \mathbb{P}} E_p$  is a compact subset of  $\mathcal{A}_f(\mathbb{Q})$  such that, for each  $p \in \mathbb{P}$ ,  $E_p$  is infinite and, for each  $n \geq 0$ ,  $\#(E \pmod{p}) > n$  for almost all  $p \in \mathbb{P}$ . Then, there exist regular bases, in particular regular bases formed by polynomials in  $\mathbb{Q}[x]$ , of the  $\widehat{\mathbb{Z}}$ -module  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\underline{E}, \widehat{\mathbb{Z}})$ . And, for any regular basis  $\{\underline{g}_n(x)\}_{n \geq 0}$ , every  $\varphi \in \mathcal{C}(\underline{E}, \mathcal{A}_f(\mathbb{Q})) \cap \prod_{p \in \mathbb{P}} \mathcal{C}(E_p, \mathbb{Q}_p)$  may be uniquely written as a uniformly convergent series in the  $\underline{g}_n$ 's with coefficients in  $\mathcal{A}_f(\mathbb{Q})$ :*

$$\varphi(x) = \sum_{n \geq 0} \underline{c}_n \underline{g}_n(x) \text{ where } \underline{c}_n \in \mathcal{A}_f(\mathbb{Q}) \text{ and } \lim_{n \rightarrow +\infty} \underline{c}_n = \underline{0}.$$

Moreover,

$$\forall p \in \mathbb{P} \quad \inf_{n \geq 0} v_p(\pi_p(\underline{c}_n)) = \inf_{\underline{y} \in \underline{E}} v_p(\pi_p(\varphi(\underline{y}))).$$

The existence of regular bases, in particular bases formed by polynomials in  $\mathbb{Q}[x]$ , follows from Theorem 5.1 and Corollary 4.6. But, we have to take care that, in general, the coefficients  $\underline{c}_n$  do not satisfy the recursive formula (14) which implied easily the uniqueness of the coefficients in the previous theorem. Nevertheless, the existence and the uniqueness of the coefficients are consequences of an extension of Theorem 1.4: following [4, Theorem 2] or [7, Theorem 2.7], the bases associated to orderings may be replaced by any basis.

Take care also that to say that  $\lim_{n \rightarrow +\infty} \underline{c}_n = \underline{0}$  in  $\mathcal{A}_f(\mathbb{Q})$  means not only that, for every  $p \in \mathbb{P}$ ,  $\lim_n \pi_p(\underline{c}_n) = 0$  in  $\mathbb{Q}_p$ , but also that there exist  $N$  such that  $\underline{c}_n \in \widehat{\mathbb{Z}}$  for  $n \geq N$ . The fact that the series  $\sum_n \underline{c}_n \underline{g}_n(x)$  converges uniformly in  $\mathcal{C}(\underline{E}, \widehat{\mathbb{Z}})$  is an obvious consequence of both conditions satisfied by the  $\underline{c}_n$ 's.

*Remark 6.4.* In Proposition 6.1, the functions of  $\prod_{p \in \mathbb{P}} \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$  were expanded as series in the binomial polynomials  $\binom{x}{n}$ . Note that these  $\binom{x}{n}$ 's form a regular basis of the  $\widehat{\mathbb{Z}}$ -module  $\text{Int}_{\mathcal{A}_f(\mathbb{Q})}(\widehat{\mathbb{Z}}, \widehat{\mathbb{Z}})$  which corresponds either to an adelic ordering of  $\widehat{\mathbb{Z}}$  as in Theorem 6.2, or to polynomials in  $\text{Int}_{\mathbb{Q}}(\widehat{\mathbb{Z}}, \widehat{\mathbb{Z}})$  as in Theorem 6.3.

## REFERENCES

- [1] Y. AMICE, Interpolation  $p$ -adique, *Bull. Soc. Math. France* **92** (1964), 117–180.
- [2] M. BHARGAVA,  $P$ -orderings and polynomial functions on arbitrary subsets of Dedekind rings, *J. reine angew. Math.* **490** (1997), 101–127.
- [3] M. BHARGAVA, The factorial function and generalizations, *Amer. Math. Monthly* **107** (2000), 783–799.
- [4] M. BHARGAVA AND K.S. KEDLAYA, Continuous functions on compact subsets of local fields, *Acta Arith.* **91** (1999), 191–198.
- [5] N. BOURBAKI, *Algèbre commutative*, Hermann, Paris, 1961.
- [6] P.-J. CAHEN AND J.-L. CHABERT, *Integer-Valued Polynomials*, Amer. Math. Soc. Surveys and Monographs, **48**, Providence, 1997.
- [7] P.-J. CAHEN AND J.-L. CHABERT, On the ultrametric Stone-Weierstrass theorem and Mahler's expansion, *Journal de Théorie des Nombres de Bordeaux*, **14** (2002), 43–57.

- [8] J.-L. CHABERT AND G. PERUGINELLI, Polynomial overrings of  $\text{Int}(\mathbb{Z})$ , to appear in *J. Commut. Algebra* (2015), arXiv: 1503.0603.
- [9] J. DIEUDONNÉ, Sur les fonctions continues  $p$ -adiques, *Bull. Sci. Math.*, 2ème série. **68** (1944), 79–95.
- [10] I. KAPLANSKY, The Weierstrass theorem in fields with valuations, *Proc. Amer. Math. Soc.* **1** (1950), 356–357.
- [11] K. MAHLER, An Interpolation Series for Continuous Functions of a  $p$ -adic Variable, *J. reine angew. Math.* **199** (1958), 23–34 and **208** (1961), 70–72.
- [12] G. PÓLYA, Über ganzwertige Polynome in algebraischen Zahlkörpern, *J. Reine Angew. Math.* **149** (1919), 97–116.

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